Using trig identities to prove the equation

Proving trig identities can be intimidating, mostly because it's hard to know where to start, or what part of the identity to work on next.

Practice definitely helps, so in this lesson we'll prove some trig equations using the trig identities that we've already learned.

Example

Prove the trig equation.

$$\cot\left(\frac{\theta}{2}\right) = \frac{\sin\theta}{1-\cos\theta}$$

We'll start with the expression on the left-hand side and derive the expression on the right-hand side from it. By the definition of cotangent, we get

$$\cot\left(\frac{\theta}{2}\right) = \frac{\cos\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$$

Multiplying top and bottom by $2\sin(\theta/2)$ gives

$$\cot\left(\frac{\theta}{2}\right) = \frac{\cos\left(\frac{\theta}{2}\right)\left[2\sin\left(\frac{\theta}{2}\right)\right]}{\sin\left(\frac{\theta}{2}\right)\left[2\sin\left(\frac{\theta}{2}\right)\right]}$$

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By the double-angle identity for sine, applied to the angle $\theta/2$, we get

$$\sin\left[2\left(\frac{\theta}{2}\right)\right] = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$$

The expression on the right-hand side of this equation is equal to the expression in our numerator, so

$$\cot\left(\frac{\theta}{2}\right) = \frac{\sin\left[2\left(\frac{\theta}{2}\right)\right]}{\sin\left(\frac{\theta}{2}\right)\left[2\sin\left(\frac{\theta}{2}\right)\right]}$$

$$\cot\left(\frac{\theta}{2}\right) = \frac{\sin\theta}{\sin\left(\frac{\theta}{2}\right)\left[2\sin\left(\frac{\theta}{2}\right)\right]}$$

$$\cot\left(\frac{\theta}{2}\right) = \frac{\sin\theta}{2\sin^2\left(\frac{\theta}{2}\right)}$$

By the half-angle identity for sine,

$$\sin^2\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{2}$$

Therefore,

$$\cot\left(\frac{\theta}{2}\right) = \frac{\sin\theta}{2\left(\frac{1-\cos\theta}{2}\right)}$$

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Canceling the 2's gives us the original equation.

$$\cot\left(\frac{\theta}{2}\right) = \frac{\sin\theta}{1-\cos\theta}$$

Let's do another example.

Example

Prove the trig equation.

$$\cos(3\theta) = \cos\theta(1 - 4\sin^2\theta)$$

Thinking of 3θ as $2\theta + \theta$ and applying the sum identity for cosine, we get

$$\cos(3\theta) = \cos(2\theta + \theta) = \cos(2\theta)(\cos\theta) - \sin(2\theta)(\sin\theta)$$

By the double-angle identity for cosine,

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

Therefore,

$$\cos(3\theta) = (\cos^2\theta - \sin^2\theta)(\cos\theta) - \sin(2\theta)(\sin\theta)$$

By the double-angle identity for sine,

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

So

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$$\cos(3\theta) = (\cos^2\theta - \sin^2\theta)(\cos\theta) - (2\sin\theta\cos\theta)(\sin\theta)$$

We can see that $\cos\theta$ is a common factor, so we'll factor it out.

$$\cos(3\theta) = (\cos\theta)[(\cos^2\theta - \sin^2\theta) - 2\sin\theta(\sin\theta)]$$

$$\cos(3\theta) = (\cos\theta)[(\cos^2\theta - \sin^2\theta) - 2\sin^2\theta]$$

$$\cos(3\theta) = (\cos\theta)[\cos^2\theta + (-\sin^2\theta - 2\sin^2\theta)]$$

$$\cos(3\theta) = (\cos\theta)(\cos^2\theta - 3\sin^2\theta)$$

By the basic Pythagorean identity, we can express $\cos^2\theta$ as $1 - \sin^2\theta$, so

$$\cos(3\theta) = (\cos\theta)[(1 - \sin^2\theta) - 3\sin^2\theta]$$

$$\cos(3\theta) = (\cos\theta)[1 + (-\sin^2\theta - 3\sin^2\theta)]$$

$$\cos(3\theta) = (\cos\theta)(1 - 4\sin^2\theta)$$

This is the equation we started with, so we've proven the trig equation.

Let's do one more.

Example

Prove the trig equation.

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$$

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We'll start by working on the left side of the equation. By the definition of tangent,

$$\tan\left(\frac{\pi}{2}-\theta\right)=\frac{\sin\left(\frac{\pi}{2}-\theta\right)}{\cos\left(\frac{\pi}{2}-\theta\right)}$$

By the difference identity for sine,

$$\sin\left(\frac{\pi}{2}-\theta\right)=\sin\left(\frac{\pi}{2}\right)(\cos\theta)-\cos\left(\frac{\pi}{2}\right)(\sin\theta)$$

Therefore,

$$\tan\left(\frac{\pi}{2}-\theta\right)=\frac{\sin\left(\frac{\pi}{2}\right)(\cos\theta)-\cos\left(\frac{\pi}{2}\right)(\sin\theta)}{\cos\left(\frac{\pi}{2}-\theta\right)}$$

By the difference identity for cosine,

$$\cos\left(\frac{\pi}{2}-\theta\right)=\cos\left(\frac{\pi}{2}\right)(\cos\theta)+\sin\left(\frac{\pi}{2}\right)(\sin\theta)$$

Substituting this result, we find that

$$\tan\left(\frac{\pi}{2}-\theta\right)=\frac{\sin\left(\frac{\pi}{2}\right)(\cos\theta)-\cos\left(\frac{\pi}{2}\right)(\sin\theta)}{\cos\left(\frac{\pi}{2}\right)(\cos\theta)+\sin\left(\frac{\pi}{2}\right)(\sin\theta)}$$

Now $\cos(\pi/2)=0$ and $\sin(\pi/2)=1$, so

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$$\tan\left(\frac{\pi}{2}-\theta\right)=\frac{1(\cos\theta)-0(\sin\theta)}{0(\cos\theta)+1(\sin\theta)}$$

$$\tan\left(\frac{\pi}{2}-\theta\right)=\frac{\cos\theta}{\sin\theta}$$

By the definition of cotangent

$$\tan\left(\frac{\pi}{2}-\theta\right)=\cot\theta$$

This is the equation we started with, so we've proven the trig equation.

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